ON A CLASS OF BODIES OF REVOLUTION WITH MINIMUM WAVE RESISTANCE

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We shall investigate the problem of constructing the generatrix AB (Fig. 1) of a body of revolution, which ensures minimum wave resistance of the body of revolution when the velocity w_∞ of the uniform oncoming flow and the coordinates of the points A and B are given. We shall study only the case for which the shock wave AC is attached. Let BC be a characteristic of the second family and CD a characteristic of the first family.

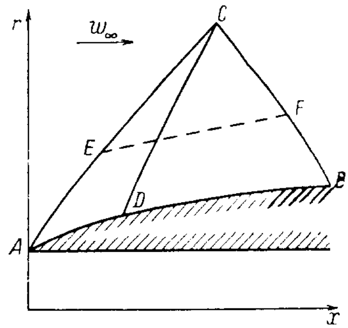

Fig. 1.

The gas flow is determined by the equations

$$\frac{\partial r\rho w \cos\vartheta}{\partial x} + \frac{\partial r\rho w \sin\vartheta}{\partial r} = 0 \tag{1}$$

$$\frac{\partial}{\partial x} r\,(p + \rho w^2 \cos^2\vartheta) + \frac{\partial}{\partial r} r\rho w^2 \sin\vartheta \cos\vartheta = 0 \tag{2}$$

$$\frac{w^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} = \frac{1}{2}\frac{\varkappa + 1}{\varkappa - 1}, \quad \frac{p}{\rho^\varkappa} = \varphi^{\varkappa-1}(\psi) \tag{3}$$

Here x, r are Cartesian coordinates in the meridional plane of the flow; w is the velocity referred to the critical flow velocity a_*; ϑ is the angle of inclination of the velocity to the flow axis x; ρ is the gas density referred to the density of the oncoming flow ρ_∞; p is the pressure referred to $\rho_\infty a_*^2$; κ is the adiabatic index; and ψ is the stream function, where

$$d\psi = r\rho w\,(\cos\vartheta\, dr - \sin\vartheta dx)$$

The solution of the formulated problem is given in [1] for the case $r_B \leqslant r_A$ and for the flow in the region DCB for the case $r_B > r_A$ if the functions obey the Euler equations on BC.

The unknown functions satisfy the following equations on BC:

$$\begin{gathered}\lambda(\varkappa\sin 2\vartheta+\sin 2\alpha)+\varkappa\mu(1-\cos 2\vartheta)=0\\ \lambda\varphi A(\alpha)\cos\alpha-\sqrt{\varkappa}\,ra(\alpha)\sin^2\vartheta=0\\ \sqrt{\varkappa r}\,dr\,/\,d\psi+\varphi A(\alpha)\sin(\vartheta-\alpha)=0\\ \sqrt{\varkappa}r^2d\mu\,/\,d\psi+\varphi A(\alpha)\left[\lambda\cos(\vartheta-\alpha)+\mu\sin(\vartheta-\alpha)\right]=0\end{gathered}\tag{4}$$

Here λ is a constant and $\mu(\psi)$ is a variable Lagrangian multiplier; α is the Mach angle, where $\rho w^2 \sin^2 \alpha = kp$

$$A(\alpha)=\left(\frac{\varkappa+1}{2\varkappa}\,\frac{1-\cos 2\alpha}{\varkappa-\cos 2\alpha}\right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}},\qquad a(\alpha)=\sqrt{\frac{\varkappa+1}{\varkappa-\cos 2\alpha}}$$

$$\alpha=\alpha(\psi),\quad \vartheta=\vartheta(\psi),\quad r=r(\psi)$$

We shall designate by χ the value divided by 2π of the wave resistance of the body of revolution with the generatrix AB. The quantity χ is expressed with the aid of Equation (2) as a contour integral along AC and BC. Subsequently we shall investigate ϕ as a function of the angle of inclination of the shock-wave to the x-axis, which we shall designate by $\sigma(\psi)$.

In searching for the contour AB the following variational problem arises. For the given constants w_∞, r_A, r_B, $X = x_B - x_A$ find the function $\sigma(\psi)$ which realizes the extremum of the function

$$\chi=\int_{\psi=\psi_A}^{\psi_C}\left\{\frac{\varkappa+1}{2\varkappa}\left(w_\infty+\frac{1}{w_\infty}\right)-a(\alpha)\left[\cos\vartheta-\frac{1}{\varkappa}\sin\alpha\sin(\vartheta-\alpha)\right]\right\}d\psi\tag{5}$$

under the isoperimetric condition

$$X=\int_{\psi=\psi_A}^{\psi_C}\left[\frac{\cot\sigma}{\sqrt{2w_\infty\psi}}+\frac{1}{\sqrt{\varkappa}\,r}\;\varphi(\sigma)A(\alpha)\cos(\vartheta-\alpha)\right]d\psi\tag{6}$$

for the conditions (4) which relate the functions α, ϑ, r, μ with the unknown function σ and for the well-known relations at the shock

$$\alpha_C=\alpha(\sigma_C),\qquad \vartheta_C=\vartheta(\sigma_C),\qquad 2\psi_C=w_\infty r_C{}^2$$

The statement of the problem considered, with the requirement that the first two equalities be satisfied, is not, of course, the only one possible. Continuous functions $\sigma(\psi)$ which satisfy the properties [2] of solutions of the system of equations (1) to (3) are possible.

We note that [3] is devoted to this same problem.

The formulated problem is degenerate. Nevertheless, we shall attempt to solve it by the methods of classical variational calculus. We shall introduce the new independent variable z through the formula $z^2 = 2w_\infty \psi$. For determining the functions $\alpha(z)$, $\vartheta(z)$, $\sigma(z)$, $r(z)$, $\mu(z)$ for $z_A \leqslant z \leqslant z_C$ we obtain the following system of Eulerian equations:

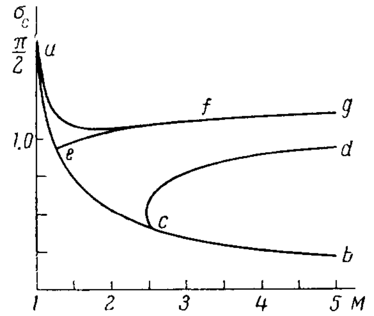

Fig. 2.

$$\frac{d\mu}{dz} = \frac{\lambda}{w_\infty r \varphi_\sigma' \sin^2 \sigma} = 0 \qquad \left(\varphi_\sigma' = \frac{1}{\varphi}\frac{\partial \varphi}{\partial \sigma}\right) \tag{7}$$

$$\frac{dr}{dz} + \frac{\lambda}{w_\infty \varphi_\sigma' [\lambda \cot(\vartheta - \alpha) + \mu] \sin^2 \sigma} = 0 \tag{8}$$

$$\lambda(\varkappa \sin 2\vartheta + \sin 2\alpha) + \varkappa\mu(1 - \cos 2\vartheta) = 0 \tag{9}$$

$$\sqrt{\varkappa}\, a(\alpha)\, r \sin^2 \vartheta = \lambda A(\alpha) \varphi \cos \alpha \tag{10}$$

$$\varphi_\sigma' z a(\alpha) [\lambda \cos(\vartheta - \alpha) + \mu \sin(\vartheta - \alpha)] \sin^2 \sigma \sin^2 \vartheta = \lambda^2 \cos \alpha \tag{11}$$

The functions α, ϑ, σ, r, μ must satisfy the isoperimetric condition (6), the boundary conditions

$$\alpha_C = \alpha(\sigma_C), \ \vartheta_C = \vartheta(\sigma_C), \ z_C = w_\infty r_C, \ r(w_\infty r_A) = r_B \tag{12}$$

where r_B is a given quantity, and the tranversality condition at $z = z_C$

$$\frac{\varkappa+1}{2\varkappa}\left(w_\infty + \frac{1}{w_\infty}\right) - a(\alpha)\left[\cos\vartheta - \frac{1}{\varkappa}\sin\alpha \sin(\vartheta - \alpha)\right] + \lambda\left[\frac{\cot\sigma}{\sqrt{2w_\infty \psi}} + \frac{1}{\sqrt{\varkappa r}}\varphi(\sigma) A(\alpha)\cos(\vartheta - \alpha)\right] + \mu\left[\frac{1}{\sqrt{2w_\infty\psi}} - \frac{1}{\sqrt{\varkappa r}}\varphi(\sigma) A(\alpha) \sin(\vartheta - \alpha)\right] = 0 \tag{13}$$

Condition (13), as S.N. Eliseev and B.M. Kiselev have shown, is satisfied as a result of the relations at the shock-wave and the first two equations of (4).

Four arbitrary quantities occur in determining the functions: the differential equations (7) and (8) give rise to two arbitrary quantities and, in addition, the quantities λ and z_C are arbitrary. Conditions (6) and (12) give five relationships.

Thus the formulated variational problem, generally speaking, does not have a solution in the classical sense. However, for certain special relations of the quantities w_∞, $r_A : X$, $r_B : X$ such a solution is possible. We shall find these special relationships [4]. To do this we shall eliminate the quantities λ and μ from Equations (9) to (11); as a result we obtain

$$A(\alpha)\,\varphi_\sigma'(\sigma)(\sin 2\alpha + \varkappa \sin 2\vartheta)\sin(\vartheta-\alpha) +$$

$$+ 2\varkappa \sin^2\vartheta\left[\frac{\sqrt{\varkappa}\,r}{z\varphi(\sigma)\sin^2\sigma} - A(\alpha)\,\varphi_\sigma'(\sigma)\cos(\vartheta-\alpha)\right] = 0 \qquad (14)$$

A solution then is found when Equation (14) is satisfied at $z = z_C$ as a result of the relations at the shock wave.

We shall mentally substitute in Equation (14) the expression for z_C at the shock wave $z_C = w_\infty r_C$ and the functions $a(\sigma_C)$ and $\vartheta(\sigma_C)$ which express the relations at the shock-wave. Then we obtain the equation $\epsilon(w_\infty, \sigma_C) = 0$. The quantity w_∞ is a parameter in the equation $\epsilon = 0$.

The roots of the equation $\epsilon = 0$ are presented in Fig. 2, in which the quantity M_∞, the Mach number of the oncoming flow, is plotted along the abscissa instead of w_∞. One of the roots of the equation $\epsilon = 0$ is

$$\sigma_C = \arcsin\frac{1}{M_\infty}$$

(line ab). This root corresponds to the case for which $r_B \leqslant r_A$ and the shock-wave AC degenerates into a characteristic of the oncoming flow. The line ag denotes those values of σ_C for which sonic velocity is attained behind the shock-wave. For greater values of σ_C the velocities become subsonic behind the shock-wave, and the theory under consideration is not applicable in this region. Finally, the lines ef and cd give two more roots of the equation $\epsilon = 0$.

Let w_∞ be fixed. We shall take a σ_C which is a root of the equation $\epsilon = 0$ and, making use of the arbitrariness in determining the scale, we shall assign a certain value to r_C. We shall calculate $\phi(\sigma_C)$ with the aid of the relations at the shock-wave and $\phi_\sigma'(\sigma_C)$. We shall then find a_C, ϑ_C, z_C from the first equations of (12), λ from Equation (10) and $\mu(z_C)$ from Equation (9). This gives the initial conditions at $z = z_C$ for integrating the system of equations (7) to (11).

We shall integrate Equations (7) to (11) from $z = z_C$ to some $z = z_*$ such that $z_* < z_C$ and the velocity behind the shock-wave is equal to sonic velocity for $\sigma(z_*)$. Making use of the equality

$$x = \frac{1}{w_\infty}\int\limits_{z=z_*}^{z}\cot\sigma dz, \quad r = \frac{z}{w_\infty}$$

at the shock-wave and the equality

$$x = x_C - \int\limits_{z=z_C}^{z}\frac{\lambda dz}{w_\infty\,\varphi_\sigma'\,[\lambda + \mu\tan(\vartheta-\alpha)]\sin^2\sigma}$$

on the characteristic *BC*, we shall construct the shock wave *AC* and the characteristic of the second family *BC* (Fig. 1).

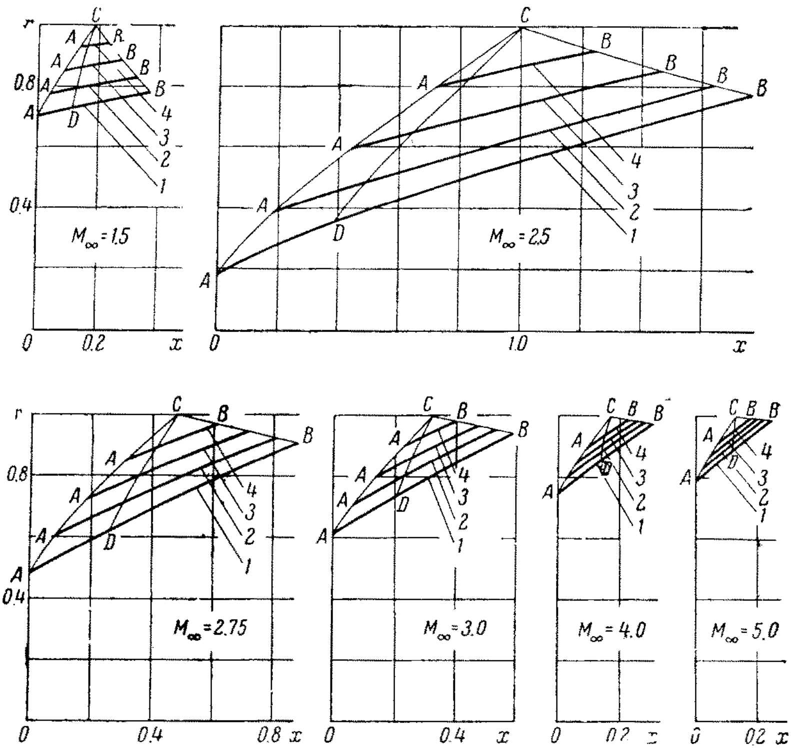

Fig. 3.

The flow, which is determined by the data given on *ACB*, can be found in the following way. The solution of the Cauchy problem for Equations (1) to (3) with the data given on *AC* permits the characteristic of the first family *CD* to be found, where $z_D = z_A$. The known characteristics *CD* and *BC* determine the solution of Equations (1) to (3), for example, within the triangle *ABC*.

All streamlines (lines of z = const or ψ = const) of the flow in the triangle *ABC* give the desired profiles. This means that if, for example, the line $\gamma(EF)$ is a streamline of the constructed flow, then of all the generatrices of bodies of revolution (at least of those near the line γ) which connect the points *E* and *F* the line γ gives precisely the least resistance. This assertion is valid because the Eulerian equations are satisfied for $z_E \leqslant z \leqslant z_C$ and all of the corresponding boundary conditions are satisfied for $z = z_C$, i.e. for the profile *EF* all of the necessary extremum conditions are satisfied.

TABLE

M_∞	Body	$r_A : X$	$r_B : X$	c_x	M_∞	Body	$r_A : X$	$r_B : X$	c_x
1,5	1	1.8796	2.0895	0.0818	3.0	1	1.0377	1.6009	0.3990
	2	2.7412	2.9387	0.0547		2	1.5157	2.0485	0.3022
	3	4.4709	4.6579	0.0325		3	2.4721	2.9805	0.2035
	4	9.6699	9.8478	0.0146		4	5.3340	5.8215	0.1026
2,5	1	0.1121	0.4433	0.2495	4.0	1	2.5188	3.2635	0.3907
	2	0.2762	0.5590	0.1934		2	3.1935	3.9190	0.3191
	3	0.5904	0.8406	0.1267		3	4.2947	5.0035	0.2463
	4	1.5492	1.7754	0.0587		4	6.3790	7.0727	0.1722
2,75	1	0.5531	1.0371	0.3881	5.0	1	3.3457	4.1783	0.3864
	2	0.8370	1.2872	0.3029		2	4.2531	5.0674	0.3135
	3	1.4257	1.8475	0.2067		3	5.7681	6.5663	0.2391
	4	3.0888	3.4871	0.1078		4	8.7588	9.5422	0.1629

The great number of streamlines of such a flow exhaust all the special solutions of the desired form since any other choice of initial data at $z = z_C$ violates the boundary conditions (12) and (13) or leads to nonfulfillment of Equations (9) to (11).

In Fig. 3 are presented examples of calculations for various M_∞. Shock-wave *AC*, characteristics of the first family *CD* and of the second family *BC* are shown. Some generatrices of bodies of revolution which have minimum resistance are presented. In the table are shown the geometrical characteristics of the points *A* and *B* and the resistance coefficients C_x referred to the area πr_B^2.

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